**Aufgabe 1:** Berechnen Sie die folgenden unbestimmten Integrale:

a) $\int \underbrace{(8x+1)}_{u}{}^4\,dx = \int u^4 \frac{1}{8}\,du = \frac{1}{8}\int u^4\,du = \frac{1}{8}\cdot\frac{1}{5}u^5 = \underline{\frac{1}{40}(8x+1)^5 + C}$

$\frac{du}{dx} = 8 \rightarrow dx = \frac{1}{8}\,du$

b) $\int \frac{1}{\sqrt[3]{2x-1}}\,dx = \frac{1}{2}\int u^{-1/3}\,du = \frac{1}{2}\cdot\frac{3}{2}u^{2/3} = \underline{\frac{3}{4}(2x-1)^{2/3} + C}$

$u = 2x-1 \quad du = 2dx$

$\rightarrow dx = \frac{1}{2}du$

c) $\int x^2 e^{10x^3}\,dx = \frac{1}{30}\int \frac{1}{\cancel{x^2}}\cancel{x^2}e^u\,du = \frac{1}{30}\int e^u\,du = \frac{1}{30}e^u + C$

$x^2\cdot e^u \quad u = 10x^3$

$du = 30x^2\,dx$

$dx = \frac{1}{30x^2}\,du$

$= \underline{\frac{1}{30}e^{10x^3} + C}$

d) $\int 5\sin(\underbrace{5x}_{u=5x})\,dx = \cancel{\frac{1}{5}}\cdot\cancel{5}\int \sin(u)\,du = -\cos(u) + C$

$du = 5dx$

$dx = \frac{1}{5}du$

$= \underline{-\cos(5x) + C}$

e) $\int \ln(\underbrace{\sin(x)}_{u=\sin(x)})\cdot\cos(x)\,dx = \int \ln(u)\cdot\cancel{\cos(x)}\cdot\frac{1}{\cancel{\cos(x)}}\,du = \int 1\cdot\ln(u)\,du$

$du = \cos(x)\,dx$

$dx = \frac{1}{\cos(x)}\,du$

↑ ↓

$= u\cdot\ln(u) - \underbrace{\int \cancel{u}\cdot\frac{1}{\cancel{u}}\,du}$

$\int du = u$

$= u\cdot\ln(u) - u + C$

$= \sin(x)\cdot\ln(\sin(x)) - \sin(x) + C$

$= \underline{\sin(x)\cdot\left(\ln(\sin(x)) - 1\right) + C}$

**Aufgabe 2:** Berechnen Sie die folgenden bestimmten Integrale:

a) $\int_0^1 \underbrace{\sqrt{2x+5}}_{u=2x+5}\,dx = \frac{1}{2}\int_5^7 u^{1/2}\,du = \frac{1}{2}\left[\frac{2}{3}u^{3/2}\right]_5^7 = \frac{1}{2}\left(\frac{2}{3}7^{3/2}\right) - \frac{1}{2}\left(\frac{2}{3}5^{3/2}\right)$

$du = 2\,dx$ $\quad x=0 \to u=5$

$dx = \frac{1}{2}du$ $\quad x=1 \to u=7$

$= \frac{1}{3}\left(7^{3/2} - 5^{3/2}\right)$

$= \frac{1}{3}\left(7\sqrt{7} - 5\sqrt{5}\right)$

b) $\int_0^\pi \sin(x)\cos^3(x)\,dx = \int_{-1}^{1} \cancel{\sin(x)}\cdot u^3\cdot\left(-\frac{1}{\cancel{\sin(x)}}\right)du = -\int_{-1}^{1} u^3\,du = -\frac{1}{4}\left[u^4\right]_{-1}^{1}$

$u = \cos(x)$

$du = -\sin(x)\,dx$ $\quad x=0 \to u=1$

$dx = -\frac{1}{\sin(x)}du$ $\quad x=\pi \to u=-1$

$= -\frac{1}{4}\underbrace{\left(1^4 - (-1)^4\right)}_{=0}$

$= 0$

c) $\int_0^\infty e^{(\ln(x)-x^2)}\,dx = \int_0^\infty \underbrace{e^{\ln(x)}}_{x}\cdot e^{-x^2}\,dx = \int_0^\infty x\,e^{-x^2}\,dx = \int_0^\infty \cancel{x}\cdot e^{-u}\cdot\frac{1}{2\cancel{x}}\,du$

$u = x^2$ $\quad du = 2x\,dx$

$dx = \frac{1}{2x}du$

$= \frac{1}{2}\int_0^\infty e^{-u}\,du = -\frac{1}{2}\left[e^{-u}\right]_0^\infty$

$= -\frac{1}{2}\underbrace{\lim_{b\to\infty} e^{-b}}_{=0} - \left(-\frac{1}{2}e^0\right) = \frac{1}{2}$

$x=0 \to u=0$

$x=\infty \to u=\infty$

d) $\int_{\cancel{0}\,1}^{\cancel{1}\,2} \frac{x^2+\frac{2}{3}x+\frac{1}{3}}{x^3+x^2+x}\,dx$

$\int_1^2 \frac{\frac{1}{3}(3x^2+2x+1)}{\underbrace{x^3+x^2+x}_{u=x^3+x^2+x}}\,dx = \int \frac{\frac{1}{3}\cancel{(\ldots)}}{u}\cdot\frac{1}{\cancel{(\ldots)}}\,du = \frac{1}{3}\int_3^{14} u^{-1}\,du$

$du = (3x^2+2x+1)\,dx$

$dx = \frac{1}{(3x^2+2x+1)}du$

$x=1 \to u=3$

$x=2 \to u=14$

$= \frac{1}{3}\left[\ln|u|\right]_3^{14}$

$= \frac{1}{3}(\ln 14 - \ln 3)$

$= \frac{1}{3}\ln\left(\frac{14}{3}\right)$

**Aufgabe 3:**

a) Zeigen Sie, dass:

$$\int_{-10}^{0} \frac{1}{3x-2}\,dx = -\frac{\ln(16)}{3}$$

$u = 3x - 2 \quad du = 3\,dx \rightarrow dx = \frac{1}{3}du$

$x = -10 \rightarrow u = -32$

$x = 0 \rightarrow u = -2$

$$\frac{1}{3}\int_{-32}^{-2} u^{-1}\,du = \frac{1}{3}\left[\ln|u|\right]_{-32}^{-2} = \frac{1}{3}\left(\ln(2) - \ln(32)\right)$$

$$= -\frac{1}{3}\left(\ln(32) - \ln(2)\right) = -\frac{1}{3}\ln\left(\frac{32}{2}\right)$$

$$= \underline{-\frac{1}{3}\ln(16)}$$

b) Zeigen Sie, dass:

$$\int \tan(x)\,dx = -\ln|\cos(x)| + C$$

Benutzen Sie dazu die Identität:

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

$$\int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{\cancel{\sin(x)}}{u}\cdot\left(-\frac{1}{\cancel{\sin(x)}}\right)du = -\int u^{-1}\,du = -\ln|u| + C$$

$$= \underline{-\ln|\cos(x)| + C}$$

$u = \cos(x)$

$du = -\sin(x)\,dx$

$dx = -\frac{1}{\sin(x)}\,du$

**Aufgabe 4:** Berechnen Sie das Integral:

$$\int_{-1}^{1} \arccos(x)\, dx$$

$$\int f^{-1}(x)\, dx = x \cdot f^{-1}(x) - F\left(f^{-1}(x)\right)$$

$$f^{-1}(x) = \arccos(x)$$
$$f(x) = \cos(x)$$
$$F(x) = \sin(x)$$

$$\int \arccos(x)\, dx = x \cdot \arccos(x) - \sin\left(\arccos(x)\right)$$

$$\cos\left(\arccos(x)\right) = x$$
$$f\left(f^{-1}(x)\right) = x$$

$$\sin^2(x) + \cos^2(x) = 1$$
$$\sin^2(x) = 1 - \cos^2(x)$$
$$\sin(x) = \sqrt{1 - \cos^2(x)}$$

$$= x \cdot \arccos(x) - \sqrt{1 - \underbrace{\cos^2(\arccos(x))}_{x^2}}$$

$$= x \cdot \arccos(x) - \sqrt{1 - x^2} + C$$

$$\int_{-1}^{+1} \arccos(x)\, dx = \left[x \cdot \arccos(x) - \sqrt{1 - x^2}\right]_{-1}^{+1}$$

$$= \left(1 \cdot \arccos(1) - \underbrace{\sqrt{\cancel{1 - 1^2}}}_{=0}\right) - \left((-1) \cdot \arccos(-1) - \underbrace{\sqrt{\cancel{1 - (-1)^2}}}_{=0}\right)$$

$$= \underbrace{\arccos(1)}_{} + \underbrace{\arccos(-1)}_{}$$

$$= 0 + \pi = \underline{\pi}$$

**Aufgabe 5:** Berechnen Sie die folgenden unbestimmten Integrale:

a) $$\int 4x\,e^x\,dx = 4x\cdot e^x - \int 4\cdot e^x\,dx = 4xe^x - 4e^x + C$$
$$= \underline{4e^x(x-1) + C}$$
$\downarrow\ \uparrow$

b) $$\int x^2 \sin(x)\,dx = x^2(-\cos(x)) - \int 2x\cdot(-\cos(x))\,dx$$
$\downarrow\ \uparrow$
$$= -x^2\cos(x) + 2\int x\cos(x)\,dx$$
$\downarrow\ \uparrow$
$$= -x^2\cos(x) + 2x\cdot\sin(x) - 2\underbrace{\int 1\cdot\sin(x)\,dx}_{(-\cos(x))}$$
$$= \cos(x)\cdot(-x^2+2) + 2x\sin(x) + C$$
$$= \underline{\cos(x)\cdot(2-x^2) + 2x\sin(x) + C}$$

c) $$\int \cos^2(x)\,dx$$
$\uparrow\downarrow$
$$= \cos(x)\cdot\sin(x) + \int \sin^2(x)\,dx$$ Trick
$$\int\cos^2(x)\,dx = \sin(x)\cdot\cos(x) + \int(1-\cos^2(x))\,dx = \sin(x)\cos(x) + \underbrace{\int dx}_{x} - \int\cos^2(x)\,dx$$
$$\sin^2(x) + \cos^2(x) = 1$$
$$\Leftrightarrow \sin^2(x) = 1 - \cos^2(x)$$
$$2\int\cos^2(x)\,dx = \sin(x)\cos(x) + x$$
$$\int\cos^2(x)\,dx = \underline{\tfrac{1}{2}(\sin(x)\cos(x) + x) + C}$$

d) $$\int x^3\ln(x)\,dx = \frac{1}{4}x^4\ln(x) - \int\frac{1}{4}x^4\cdot\frac{1}{x}\,dx$$
$\uparrow\ \downarrow$
$$= \frac{1}{4}x^4\ln(x) - \frac{1}{4}\underbrace{\int x^3\,dx}_{\frac{1}{4}x^4} = \frac{1}{4}x^4\ln(x) - \frac{1}{4}\cdot\frac{1}{4}x^4$$
$$= \frac{1}{4}x^4\left(\ln(x) - \frac{1}{4}\right)$$
$$= \underline{\frac{x^4}{16}(4\ln(x) - 1) + C}$$

e) $$\int\sqrt{x}\,\ln(x)\,dx = \frac{2}{3}x^{3/2}\cdot\ln(x) - \int\frac{2}{3}x^{3/2}\cdot\frac{1}{x}\,dx$$
$\uparrow\ \downarrow$
$$= \frac{2}{3}x^{3/2}\ln(x) - \frac{2}{3}\underbrace{\int x^{1/2}\,dx}_{\frac{2}{3}x^{3/2}} = \frac{2}{3}x^{3/2}\ln(x) - \frac{2}{3}\cdot\frac{2}{3}x^{3/2}$$
$$= \frac{2}{3}x^{3/2}\left(\ln(x) - \frac{2}{3}\right)$$
$$= \underline{\frac{2}{9}x^{3/2}(3\ln(x) - 2) + C}$$

**Aufgabe 6:** Berechnen Sie die folgenden bestimmten Integrale:

a) $\int_0^\pi 3x^2 \sin(x)\,dx = \left[3x^2\cdot(-\cos(x))\right]_0^\pi - \int_0^\pi 6x\cdot(-\cos(x))\,dx = [\ \cdots\ ] + 6\int_0^\pi x\cdot\cos(x)\,dx$

$\int_0^\pi x\cdot\cos(x)\,dx = \left[x\cdot\sin(x)\right]_0^\pi - \underbrace{\int_0^\pi 1\cdot\sin(x)\,dx}_{[-\cos(x)]_0^\pi} = \left[x\cdot\sin(x)\right]_0^\pi + \left[\cos(x)\right]_0^\pi$

$= \left[3x^2\cdot(-\cos(x))\right]_0^\pi + 6\left[x\cdot\sin(x)\right]_0^\pi + 6\left[\cos(x)\right]_0^\pi$

$= 3\pi^2\cdot\underbrace{(-\cos(\pi))}_{+1} + 6\cdot\pi\cdot\underbrace{\sin(\pi)}_{=0} + 6\cos(\pi) - 6\cos(0)$

$= 3\pi^2 - 6 - 6 = 3\pi^2 - 12 = \underline{3(\pi^2-4)}$

b) $\int_0^{2\pi} \sin^2(x)\,dx = \left[\sin(x)\cdot(-\cos(x))\right]_0^{2\pi} + \int_0^{2\pi} \cos^2(x)\,dx$

$\int_0^{2\pi} \sin^2(x)\,dx = [\ \cdots\ ] + \underbrace{\int_0^{2\pi} 1\,dx}_{[x]_0^{2\pi}} - \int_0^{2\pi} \sin^2(x)\,dx \qquad \Big| + \int_0^{2\pi}\sin^2(x)\,dx$

$\cos^2(x) = 1 - \sin^2(x)$

$\cancel{2}\int_0^{2\pi} \sin^2(x)\,dx = \frac{1}{2}\left[\underbrace{\cancel{\sin(x)\cdot(-\cos(x))}}_{=0}\right]_0^{2\pi} + \frac{1}{2}\left[x\right]_0^{2\pi} \qquad \Big| :2$

$= \frac{1}{2}(2\pi - 0) = \underline{\pi}$

c) $\int_0^\pi (x-1)^2 \cos(x)\,dx = (x-1)^2\sin(x) - \int 2(x-1)\cdot\sin(x)\,dx$

$= (x-1)^2\sin(x) + 2(x-1)\cos(x) + 2\underbrace{\int \cancel{1}\cos(x)\,dx}_{\sin(x)}$

$= (x-1)^2\sin(x) + 2(x-1)\cos(x) + 2\sin(x) + C$

$= \sin(x)\left((x-1)^2+2\right) + \cos(x)\cdot\left(2(x-1)\right) + C$

$\int_0^\pi (x-1)^2\cos(x)\,dx = \left[\underbrace{\cancel{\sin(x)(x^2-2x+3)}}_{=0} + \cos(x)(2x-2)\right]_0^\pi$

$= \cos\pi\,(2\pi-2) - \cos(0)(2\cdot 0-2)$

$= (-1)\cdot(2\pi-2) + 2 = -2\pi+2+2 = \underline{4-2\pi}$

d) $\int_{-1}^{0} (x+1)^2\, e^{-x}\, dx$ $\downarrow$ $\uparrow$

$$= -(x+1)^2 e^{-x} + \int 2(x+1)\cdot e^{-x}\, dx$$

$$= -(x+1)^2 e^{-x} - 2(x+1)\cdot e^{-x} + 2\underbrace{\int e^{-x}\, dx}_{-e^{-x}}$$

$$= -(x+1)^2 e^{-x} - 2(x+1)e^{-x} - 2e^{-x}$$

$$= e^{-x}\left(-x^2-2x-1-2x-2-2\right)$$

$$= -e^{-x}\left(x^2+4x+5\right) + C$$

$$\int_{-1}^{0} (x+1)^2 e^{-x} dx = \left[-e^{-x}\left(x^2+4x+5\right)\right]_{-1}^{0} = \left((-1)\cdot(5)\right) - (-e\cdot 2)$$

$$= -5 + 2e$$

$$= \underline{2e-5}$$

Antw: 5e)

e) $\int_{1}^{e} \sqrt{x}\, \ln(x)\, dx \overset{\downarrow}{=} \left[\frac{2}{9}x^{3/2}\left(3\ln(x)-2\right)\right]_1^e$

$$= \left(\frac{2}{9}e^{3/2}\left(3\underbrace{\ln(e)}_{=1}-2\right)\right) - \left(\frac{2}{9}\,1^{3/2}\left(3\underbrace{\ln(1)}_{=0}-2\right)\right)$$

$$= \frac{2}{9}e^{3/2} - \frac{2}{9}\cdot 1\cdot(-2)$$

$$= \underline{\frac{2}{9}\left(e^{3/2}+2\right)}$$

**Aufgabe 7:** Berechnen Sie die folgenden Integrale mit Hilfe der Partialbruchzerlegung:

a) $\int \frac{2}{x^2 - 2x}\, dx$

Ansatz: $\underbrace{\frac{A}{x} + \frac{B}{x-2}}_{} = \frac{2}{x^2-2x} = -\frac{1}{x} + \frac{1}{x-2}$

Nullstellen im Nenner: $x(x-2)$ → $x=0$, $x=2$

$\frac{A(x-2)+Bx}{x^2-2x} = \frac{2}{x^2-2x}$

$\rightarrow A(x-2) + Bx = 2 \longrightarrow x\cdot(A+B) - 2A = 0\cdot x + 2$

Koeffizientenvergleich:

$A+B=0$ $\quad -2A = 2 \longrightarrow A = -1$

$\hookrightarrow B = 1$

$\int \frac{2}{x^2-2x}\,dx = -\int \frac{1}{x}\,dx + \int \frac{1}{x-2}\,dx = -\ln|x| + \ln|x-2| + C$

b) $\int \frac{x-9}{x^2+3x}\, dx$

NS: $x(x+3)$

$\frac{x-9}{x^2+3x} = \frac{A}{x} + \frac{B}{x+3} = \frac{A(x+3)+Bx}{x(x+3)}$

$A(x+3) + Bx = x - 9$

$x\cdot(A+B) + 3A = 1\cdot x - 9$

$A+B = 1$ $\quad 3A = -9 \rightarrow A = -3$

$B = 4$

$\int \frac{x-9}{x^2+3x}\,dx = -\int \frac{3}{x}\,dx + \int \frac{4}{x+3}\,dx = -3\ln|x| + 4\ln|x+3| + C$

c) $\int \frac{1}{x^2+8x-9}\, dx$

NS: $(x+9)(x-1)$

$\frac{1}{x^2+8x-9} = \frac{A}{x+9} + \frac{B}{x-1} = \frac{A(x-1)+B(x+9)}{x^2+8x-9}$

$A(x-1) + B(x+9) = 1$

$x\cdot(A+B) - A + 9B = 0\cdot x + 1$

$A+B = 0$ $\quad 9B - A = 1 \longrightarrow A = 9B - 1$

$(9B-1) + B = 0$

$10B = 1 \rightarrow B = \frac{1}{10}$ $\qquad A = 9\cdot\frac{1}{10} - 1 = \frac{9-10}{10} = -\frac{1}{10}$

$\int \frac{1}{x^2+8x-9}\,dx = -\frac{1}{10}\int \frac{1}{x+9}\,dx + \frac{1}{10}\int \frac{1}{x-1}\,dx$

$= -\frac{1}{10}\ln|x+9| + \frac{1}{10}\ln|x-1| + C$

$= \frac{1}{10}\left(\ln|x-1| - \ln|x+9|\right) + C$

d) $\displaystyle\int \frac{x+2}{(x-2)^2}\,dx$

NS: $(x-2)(x-2)$

$$\frac{x+2}{(x-2)^2} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} = \frac{A(x-2)+B}{(x-2)^2}$$

$$A(x-2) + B = x+2$$

$$x\cdot A + (B-2A) = 1\cdot x + 2$$

$A = 1$ $\qquad B - 2\cdot 1 = 2$

$\qquad B = 2+2 = 4$

$$\int \frac{x+2}{(x-2)^2}\,dx = \int \frac{1}{x-2}\,dx + \int \frac{4}{(x-2)^2}\,dx = \ln|x-2| + 4\int u^{-2}\,du$$

$u = x-2$

$du = dx$

$$= \ln|x-2| + 4(-u^{-1})$$

$$= \ln|x-2| - \frac{4}{x-2} + C$$

e) $\displaystyle\int \frac{4-10x^2}{x-x^2}\,dx = \int \frac{10x^2-4}{x^2-x}\,dx = \int\left(10 + \frac{10x-4}{x^2-x}\right)dx$

$$(10x^2-4) : (x^2-x) = 10 + \frac{10x-4}{x^2-x}$$

$-10x^2+10x$

$0 \ +10x-4$

$$\frac{10x-4}{x^2-x} = \frac{A}{x} + \frac{B}{x-1} = \frac{A(x-1)+Bx}{x(x-1)}$$

NS: $x(x-1)$

$$x\cdot(A+B) - A = 10x - 4$$

$A+B = 10$ $\qquad -A = -4 \rightarrow A = 4$

$\hookrightarrow B = 6$

$$= \int 10\,dx + \int \frac{4}{x}\,dx + \int \frac{6}{x-1}\,dx$$

$$= 10x + 4\ln|x| + 6\ln|x-1| + C$$