Aufgabe 1: Bestimmen Sie von den folgenden Funktionen die erste Ableitung.

a) $g(u) = \sqrt[3]{u^2 - 4 + 2xu}$ $= \sqrt[3]{f(u)} = f^{\frac{1}{3}}(u) \rightarrow \frac{1}{3} f^{-\frac{2}{3}}(u) \cdot f'(u) = \frac{1}{3} \cdot (u^2 - 4 + 2xu)^{-\frac{2}{3}} \cdot 2(u+x)$

$f(u) = u^2 - 4 + 2xu$

$f'(u) = 2u - 0 + 2x = 2(u+x)$

$= \underline{\frac{2(u+x)}{3 \cdot \sqrt[3]{(u^2-4+2xu)^2}}} = g'(u)$

b) $f(x) = \sqrt{\cos(x)}$ $= \sqrt{g(x)}$ $\quad f'(x) = \frac{1}{2} \cdot g^{-\frac{1}{2}}(x) \cdot g'(x) = \frac{g'(x)}{2\sqrt{g(x)}} = \underline{-\frac{\sin(x)}{2\sqrt{\cos(x)}}}$

$g(x) = \cos(x)$

$g'(x) = -\sin(x)$

c) $f(x) = \dfrac{\sin(ax)}{\cos(bx^2)}$ $= \dfrac{u(x)}{v(x)}$ $\quad u(x) = \sin(ax) \rightarrow u'(x) = \cos(ax) \cdot a = a\cos(ax)$

$v(x) = \cos(bx^2) \rightarrow v'(x) = -\sin(bx^2) \cdot 2bx = -2bx\sin(bx^2)$

$f'(x) = \dfrac{u'v - uv'}{v^2} = \underline{\dfrac{a\cos(ax) \cdot \cos(bx^2) + \sin(ax) \cdot 2bx\sin(bx^2)}{\cos^2(bx^2)}}$

d) $h(x) = \ln\left(\frac{x^2}{e^x}\right)$ $= \ln(a(x)) \longrightarrow h'(x) = \frac{1}{a(x)} \cdot a'(x) = \frac{a'(x)}{a(x)}$

$a(x) = \frac{x^2}{e^x} \longrightarrow a'(x) = \frac{2x \cdot e^x - x^2 \cdot e^x}{e^{2x}} = \frac{(2x - x^2)\cancel{e^x}}{e^x \cdot \cancel{e^x}} = \frac{x(2-x)}{e^x}$

$h'(x) = \frac{\cancel{x}(2-x) \cdot \cancel{e^x}}{\cancel{e^x} \cdot x^2} = \underline{\frac{2-x}{x}}$

e) $f(p) = p^3 \cdot \sin(p^2 x)$ $\quad f'(p) = u'v + uv' = 3p^2 \cdot \sin(p^2x) + p^3 \cdot \cos(p^2x) \cdot 2px$

$= \underline{3p^2 \sin(p^2x) + 2p^4 x \cos(p^2x)}$

$u(p) \quad v(p)$

$u'(p) = 3p^2 \quad v'(p) = \cos(p^2x) \cdot 2px$

Aufgabe 2: Bestimmen Sie von den folgenden Funktionen die erste und die zweite Ableitung.

a) $f(x) = (x^2-1)(2x^4+2)$

$u(x) = x^2-1 \rightarrow u'(x) = 2x$
$v(x) = 2x^4+2 \rightarrow v'(x) = 8x^3$

$f'(x) = u'v+uv' = 2x\cdot(2x^4+2) + (x^2-1)\cdot 8x^3 = 4x^5+4x+8x^5-8x^3$
$= \underline{12x^5-8x^3+4x}$

$f''(x) = \underline{60x^4-24x^2+4}$

b) $f(x) = -4x \cdot cos(x)$

$u(x) = -4x \rightarrow u'(x) = -4$
$v(x) = \cos(x) \rightarrow v'(x) = -\sin(x)$

$f'(x) = u'v+uv' = -4\cdot\cos(x) + (-4x)\cdot(-\sin(x)) = \underline{-4\cos x + 4x\sin x}$

$f''(x) = 4\sin x + 4\sin x + 4x\cos x = \underline{8\sin x + 4x\cos x}$

c) $f(x) = (x+9)\cdot\sqrt{x}$

$u(x) = x+9 \rightarrow u'(x) = 1$
$v(x) = x^{1/2} \rightarrow v'(x) = \frac{1}{2}x^{-1/2}$

$f'(x) = u'v+uv' = 1\cdot x^{1/2} + (x+9)\cdot\frac{1}{2}x^{-1/2} = \sqrt{x} + \frac{x+9}{2\sqrt{x}} = \frac{\overbrace{2x+x}^{3x}+9}{2\sqrt{x}} = \underline{\frac{3(x+3)}{2\sqrt{x}}}$

$f''(x) = \frac{3\cdot2\cdot\sqrt{x} - 3(x+3)\cdot\frac{1}{\sqrt{x}}}{4x} = \frac{6x-3(x+3)}{4x\sqrt{x}} = \underline{\frac{3(x-3)}{4x\sqrt{x}}}$

d) $f(x) = \dfrac{\tan(x)}{\sin(x)} = \frac{\cancel{\sin(x)}}{\cos(x)\cdot\cancel{\sin(x)}} = \frac{1}{\cos(x)}$

$\tan(x) = \frac{\sin(x)}{\cos(x)}$

$f'(x) = -u^{-2}\cdot u' = \frac{-u'}{u^2} = \frac{+\sin(x)}{\cos^2(x)}$

$u = \cos(x)$
$u' = -\sin(x)$

$= \frac{\sin(x)}{\cos(x)\cdot\cos(x)} = \underline{\frac{\tan(x)}{\cos(x)}}$

$f''(x) = \frac{u'v-uv'}{v^2} = \frac{\frac{1}{\cos^2 x}\cdot\cos x + \tan x\cdot\sin x}{\cos^2 x} = \frac{1}{\cos^3 x} + \frac{\sin^2 x}{\cos^3 x}$

$u = \tan x \quad u' = \frac{1}{\cos^2 x}$
$v = \cos x \quad v' = -\sin x$

$= \frac{1+(1-\cos^2 x)}{\cos^3 x} = \underline{\frac{2-\cos^2(x)}{\cos^3 x}}$

e) $f(x) = \cot^2(x)$

$= \frac{1}{\tan^2(x)} = \tan^{-2}(x) = u^{-2}(x)$

$u(x) = \tan(x) \rightarrow u'(x) = \frac{1}{\cos^2 x}$

$f'(x) = -2u^{-3}(x)\cdot u'(x) = -\frac{2u'(x)}{u^3(x)} = -\frac{2\cdot\frac{1}{\cos^2 x}}{\tan^3 x} = -\frac{2}{\frac{\sin^3 x\cdot\cancel{\cos^2 x}}{\cos^3 x}} = \underline{-\frac{2\cos x}{\sin^3 x}}$

$f''(x) = -\frac{-2\sin x\cdot\sin^3 x - 2\cos x\cdot 3\sin^2 x\cdot\cos x}{\sin^6 x} = \frac{2\sin^{\cancel{4}2} x + 6\cancel{\sin^2 x}\cos^2 x}{\sin^{\cancel{6}4} x}$

$= \frac{2\sin^2 x + 6\cos^2 x}{\sin^4 x}$

$= \frac{2\sin^2 x + 6 - 6\sin^2 x}{\sin^4 x}$

$= \underline{\frac{6-4\sin^2 x}{\sin^4 K}}$

Aufgabe 3: Bestimmen Sie die x-Werte der Maxima, Minima und Wendepunkte der folgenden Funktionen.

a) $f(x) = \frac{2}{3}x^3 + x^2 - 24x + 3$

$f'(x) = 2x^2 + 2x - 24$ $\quad f'(x) \overset{!}{=} 0$ $\quad 2x^2 + 2x - 24 = 0 \quad |:2$

$f''(x) = 4x + 2$ $\quad f''(x) \overset{!}{=} 0$ $\quad x^2 + x - 12 = 0$

$+4-3 \quad (+4)\cdot(-3)$

$4x + 2 = 0$

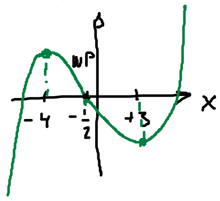
$4x = -2$

$\underline{x = -\frac{1}{2}}$ (WP)

$(x+4)(x-3) = 0 \rightarrow \underline{x = -4}$ (~~Extr.~~) Max.
$\underline{x = +3}$ (~~Extr.~~) Min.

$f''(-4) = 4\cdot(-4)+2 < 0 \rightarrow$ Maximum

$f''(+3) = 4\cdot 3 + 2 > 0 \rightarrow$ Minimum

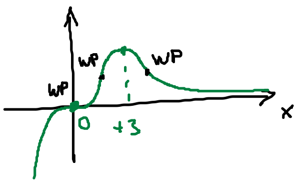


b) $f(x) = x^3 e^{-x}$

$f'(x) = 3x^2 e^{-x} + x^3\cdot(-e^{-x}) = (3x^2 - x^3)e^{-x}$

$f'(x) \overset{!}{=} 0 \rightarrow x^2(3-x) \overset{!}{=} 0 \rightarrow \underline{x = 0}, \; \underline{x = 3}$ (~~Extr.~~) $\rightarrow$ Maximum

$\hookrightarrow$ Sattelpunkt



$f''(x) = (6x - 3x^2)e^{-x} + (3x^2 - x^3)\cdot(-e^{-x})$

$= (6x - 3x^2 - 3x^2 + x^3)e^{-x}$

$= (x^3 - 6x^2 + 6x)e^{-x} = (x^2 - 6x + 6)xe^{-x}$

$a = 1, \; b = -6, \; c = 6$

$f''(x) \overset{!}{=} 0 \quad \underline{x = 0}$, (WP)

$D = b^2 - 4ac = 36 - 4\cdot 1\cdot 6 = 12$

$\sqrt{D} = \sqrt{12} = 2\sqrt{3}$

$f''(0) = 0$

$f''(3) = (-3)\cdot 3\cdot e^{-3} < 0$

$\hookrightarrow$ Maximum

$\rightarrow \underline{x = 3 \pm \sqrt{3}}$ (WP)

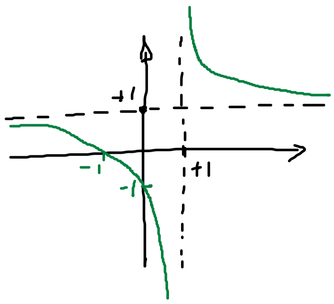
c) $f(x) = \frac{x+1}{x-1}$

$f'(x) = \frac{1\cdot(x-1) - (x+1)\cdot 1}{(x-1)^2} = \frac{-2}{(x-1)^2}$

$f'(x) \overset{!}{=} 0 \rightarrow$ keine Extrema

$f''(x) = \frac{d}{dx}\left(-2\cdot(x-1)^{-2}\right) = 4(x-1)^{-3}\cdot 1 = \frac{4}{(x-1)^3}$

$f''(x) \overset{!}{=} 0 \rightarrow$ keine WP



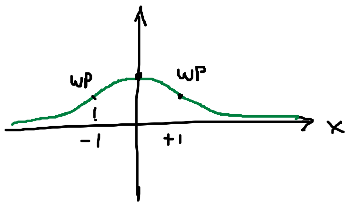
d) $f(x) = \sqrt{e^{-x^2}}$

$$f'(x) = \frac{d}{dx}\left(u^{1/2}(x)\right) = \frac{1}{2}u^{-1/2}(x)\cdot u'(x) = \frac{u'(x)}{2u^{1/2}(x)} = \frac{-\cancel{2}xe^{-x^2}}{\cancel{2}\sqrt{e^{-x^2}}}$$

$u(x) = e^{-x^2}$

$u'(x) = e^{-x^2}\cdot(-2x) = -2xe^{-x^2}$

$$= \underline{-x\sqrt{e^{-x^2}}}$$



$f'(x) \overset{!}{=} 0 \quad \rightarrow \quad x = 0 \quad$ Extr. Max.

$$f''(x) = (-1)\cdot\sqrt{e^{-x^2}} + (-x)\cdot\left(-x\sqrt{e^{-x^2}}\right)$$

$$= -\sqrt{e^{-x^2}} + x^2\sqrt{e^{-x^2}} = \underline{\sqrt{e^{-x^2}}\cdot(x^2-1)}$$

$f''(x) \overset{!}{=} 0 \quad \rightarrow \quad \underline{x = \pm 1}$ (WP)

$f''(0) = -1 < 0 \quad \rightarrow \quad$ Maximum

e) $f(x) = \dfrac{x^2-9}{x+1} = \dfrac{(x-3)(x+3)}{(x+1)}$

$$f'(x) = \frac{(2x)\cdot(x+1) - (x^2-9)\cdot 1}{(x+1)^2} = \frac{2x^2+2x-x^2+9}{(x+1)^2} = \underline{\frac{x^2+2x+9}{(x+1)^2}}$$

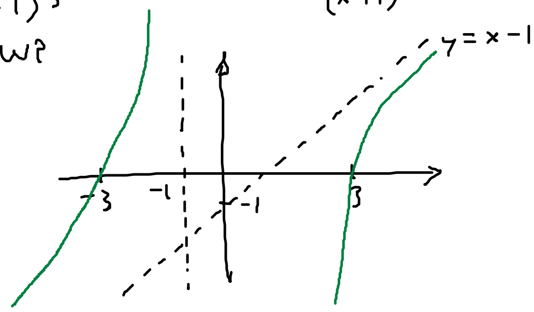
$$f'(x) \overset{!}{=} 0 \qquad \underbrace{x^2+2x+1}_{(x+1)^2}\ \underbrace{-1+9}_{+8} \overset{!}{=} 0$$

$(x+1)^2 = -8 \quad \rightarrow$ keine Extrema!

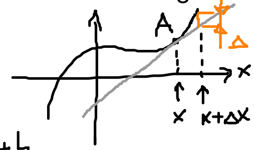
$$f''(x) = \frac{(2x+2)\cdot(x+1)^2 - (x^2+2x+9)\cdot 2(x+1)\cdot 1}{(x+1)^4} = \frac{2\cancel{(x+1)}\left[(x+1)(x+1) - x^2-2x-9\right]}{(x+1)^{\cancel{4}\,3}}$$

$$= \frac{2\cdot\left[\cancel{x^2}+\cancel{2x}+1-\cancel{x^2}-\cancel{2x}-9\right]}{(x+1)^3} = -\frac{16}{(x+1)^3}$$

$f''(x) \overset{!}{=} 0 \quad \rightarrow \quad$ keine WP



Aufgabe 4: Stellen Sie zuerst die Gleichung der Tangente auf, die die Funktion f an der bestreffenden Stelle berührt. Die Tangente ist eine lineare Näherung unserer Funktion um die betreffende Stelle herum. Ermitteln Sie dann die prozentuale Abweichung der Näherung von der Funktion für $\Delta x = 0.1$.



a) $f(x) = \frac{1}{4}x^2$ in $x = 3$

$f'(x) = \frac{1}{2}x$ Tangente: $y(x) = mx + b$ $\longrightarrow y(x) = \frac{3}{2}x - \frac{9}{4}$

$f(x=3) = \frac{1}{4}3^2 = \frac{9}{4} \rightarrow A(3, \frac{9}{4})$

Steigung: $m = \frac{1}{2} \cdot 3 = \frac{3}{2}$ $\quad \frac{9}{4} = \frac{3}{2} \cdot 3 + b \rightarrow b = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}$

$\Delta = f(x+\Delta x) - y(x+\Delta x) = \underbrace{\frac{1}{4}(3.1)^2}_{2.4025} - \frac{3}{2}(3.1) + \frac{9}{4} = 0.0025$

$\frac{\Delta}{f} = \frac{0.0025}{2.4025} = \underline{0.10\,\%}$

b) $f(x) = \sin\left(\frac{x}{2}\right)$ in $x = \frac{\pi}{2}$ $\quad f(\frac{\pi}{2}) = \sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} \rightarrow A(\frac{\pi}{2}, \frac{1}{\sqrt{2}})$

$f'(x) = \cos(\frac{x}{2}) \cdot \frac{1}{2}$ $\quad f'(x = \frac{\pi}{2}) = \cos(\frac{\pi}{4}) \cdot \frac{1}{2} = \frac{1}{2\sqrt{2}}$

$y(x = \frac{\pi}{2}) = \frac{1}{\sqrt{2}} = \underbrace{\frac{1}{2\sqrt{2}} \cdot \frac{\pi}{2}}_{m \cdot x} + b \longrightarrow b = \frac{1}{\sqrt{2}} - \frac{\pi}{2} \cdot \frac{1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)$

$\rightarrow y(x) = \frac{1}{2\sqrt{2}} \cdot x + \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)$ $\quad \Delta = \sin\left(\frac{\frac{\pi}{2} + 0.1}{2}\right) - \frac{1}{2\sqrt{2}} \cdot \left(\frac{\pi}{2} + 0.1\right) - \frac{1}{\sqrt{2}}\left(1 - \frac{\pi}{4}\right)$

$\Delta = 0.000902 \longrightarrow \frac{\Delta}{f} = \underline{0.13\,\%}$

c) $f(x) = \sqrt{\sin(2x)}$ in $x = \frac{\pi}{4}$

$f(x) = u^{1/2} \longrightarrow f'(x) = \frac{1}{2} u^{-1/2} \cdot u'(x) = \frac{u'(x)}{2\sqrt{u(x)}} = \frac{\cancel{2}\cos(2x)}{\cancel{2}\sqrt{\sin(2x)}}$

$\uparrow$

$u(x) = \sin(2x)$

$u'(x) = \cos(2x) \cdot 2$ $\quad f'(x = \frac{\pi}{4}) = \frac{\cos(\frac{\pi}{2})}{\sqrt{\ldots}} = 0$

$f(\frac{\pi}{4}) = \sqrt{\sin(\frac{\pi}{2})} = 1 \rightarrow A(\frac{\pi}{4}, 1)$

$\underline{y(\frac{\pi}{4}) = 1}$ $\quad |\Delta| = \left|\underbrace{f(\frac{\pi}{4} + 0.1)}_{0.9900} - \underbrace{y(\frac{\pi}{4} + 0.1)}_{=1}\right| = 0.01002$

$\frac{\Delta}{f} = \underline{1{,}012\,\%}$