Erste Ableitungen

Aufgabe 1: Leiten Sie die folgenden Funktionen nach ihrem Argument ab.

a) $f(x) = x^5 + \frac{1}{4}x^4 - 3x^2 + x - 1$

$f'(x) = \underline{5x^4 + x^3 - 6x + 1}$

b) $f(x) = -\big(\sin(x) - \cos(x)\big) = -\sin(x) + \cos(x)$

$f'(x) = -\cos(x) - \sin(x) = \underline{-\left(\cos(x) + \sin(x)\right)}$

c) $f(u) = -\frac{x^2}{u} = -x^2 \cdot u^{-1}$

$f'(u) = -x^2 \cdot (-1) \cdot u^{-2} = x^2 u^{-2} = \underline{\left(\frac{x}{u}\right)^2}$

d) $f(b) = (\frac{a}{e})^b$ $\quad f(x) = c^x \longrightarrow f'(x) = \log c \cdot c^x$

$f'(b) = \log\left(\frac{a}{e}\right) \cdot \left(\frac{a}{e}\right)^b = \left[\log a - \underbrace{\log e}_{=1}\right] \cdot \left(\frac{a}{e}\right)^b = \underline{\left[\log a - 1\right] \cdot \left(\frac{a}{e}\right)^b}$

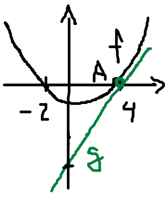
e) $f(p) = \sum_{m=1}^{3} \Big((-1)^m \cdot 4m \cdot p^m\Big) = (-1) \cdot 4 \cdot 1 \cdot p^1 + 4 \cdot 2 \cdot p^2 + (-1) \cdot 4 \cdot 3 \cdot p^3$

$= -4p + 8p^2 - 12p^3$

$f(p) = -12p^3 + 8p^2 - 4p$

$f'(p) = \underline{-36p^2 + 16p - 4}$

Aufgabe 2: Wir legen eine Tangente an die Funktion f. Bestimmen Sie zuerst den Punkt auf der Funktion und die Steigung in diesem Punkt. Stellen Sie dann die Gleichung der Tangente g auf.



a) $f(x) = \frac{1}{5}(x-4)(x+2)$ für $x = 4$

$f(4) = 0 \longrightarrow \underline{A(4,0)}$

$f(x) = \frac{1}{5}(x^2 - 2x - 8)$

$f'(x) = \frac{1}{5}(2x-2) = \frac{2}{5}(x-1)$

$f'(4) = \frac{6}{5} = 1{,}2 \longrightarrow g: y(x) = 1.2x + b$

A eingesetzt: $0 = 1.2 \cdot 4 + b$

$\longrightarrow b = -4{,}8$

$\Rightarrow g: \underline{y(x) = 1.2x - 4.8}$

b) $f(x) = 2(1-x^2)$ für $x = -1$

$f(-1) = 2(1-(-1)^2) = 2 \cdot (1-1) = 0 \Big\} \underline{A(-1,0)}$

$f(x) = 2 - 2x^2 = -2x^2 + 2$

$f'(x) = -4x \longrightarrow f'(-1) = -4 \cdot (-1) = +4 \qquad y(x) = 4x + b \Big\} g: \underline{y(x) = 4x + 4}$

A: $0 = 4 \cdot (-1) + b \longrightarrow b = 4$

c) $f(x) = \tan(x)$ für $x = 0$

$f'(x) = 1 + \tan^2(x)$

$f(0) = \tan(0) = 0 \longrightarrow \underline{A(0,0)}$

$f'(0) = 1 + \underbrace{\tan^2(0)}_{0} = 1 \longrightarrow y(x) = x + b \Big\} \underline{g: y = x}$

A: $0 = 0 + b \longrightarrow b = 0$

d) $f(x) = \frac{1}{2}\sin(x)$ für $x = \frac{\pi}{4}$

$f\left(\frac{\pi}{4}\right) = \frac{1}{2}\sin\left(\frac{\pi}{4}\right) = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \longrightarrow \underline{A\left(\frac{\pi}{4}, \frac{1}{2\sqrt{2}}\right)}$

$f'(x) = \frac{1}{2}\cos(x)$

$f'\left(\frac{\pi}{4}\right) = \frac{1}{2}\cos\left(\frac{\pi}{4}\right) = \frac{1}{2} \cdot \frac{1}{\sqrt{2}} \longrightarrow y(x) = \frac{1}{2\sqrt{2}}x + b$

A: $\frac{1}{2\sqrt{2}} = \frac{1}{2\sqrt{2}} \cdot \frac{\pi}{4} + b \quad \Big| - \frac{1}{2\sqrt{2}} \cdot \frac{\pi}{4}$

$\frac{1}{2\sqrt{2}}\left(1 - \frac{\pi}{4}\right) = b$

$\Big\} g: y(x) = \underline{\frac{1}{2\sqrt{2}} \cdot x + \frac{1}{2\sqrt{2}}\left(1 - \frac{\pi}{4}\right)}$

e) $f(x) = \cos(x) - 0.5x + 3$ für $x = 0$

$f(0) = \underbrace{\cos(0)}_{=1} - \underbrace{\cancel{0.5 \cdot 0}}_{=0} + 3 = 4 \hookrightarrow \underline{A(0,4)}$

$f'(x) = -\sin(x) - 0.5 \longrightarrow f'(0) = -\overset{=0}{\cancel{\sin(0)}} - 0.5 = -0.5 \longrightarrow y(x) = -\frac{1}{2}x + b$

A: $4 = -\underbrace{\cancel{\tfrac{1}{2} 0}}_{=0} + b \rightarrow b = 4 \longrightarrow g: \underline{y(x) = -\frac{1}{2}x + 4}$

Aufgabe 3: In welchen Punkten ist die Funktion f parallel zur Geraden g?

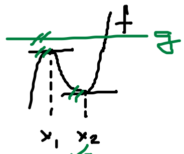
a) $f(x) = x^3 - 5x^2 + 2x + 2$ und $g(x) = 4$

$f'(x) = 3x^2 - 10x + 2$ $\qquad g'(x) = 0$

$3x^2 - 10x + 2 \overset{!}{=} 0$

$a = 3,\ b = -10,\ c = 2 \longrightarrow D = b^2 - 4ac = 100 - 4 \cdot 3 \cdot 2 = 100 - 24 = 76 \longrightarrow \sqrt{D} = \sqrt{76} = \sqrt{4 \cdot 19} = 2\sqrt{19}$

$\hookrightarrow x_{1,2} = \frac{10 \pm 2\sqrt{19}}{6} = \underline{\frac{5}{3} \pm \frac{1}{3}\sqrt{19}}$

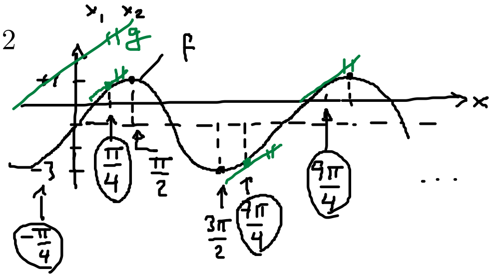


b) $f(x) = 2\sin(x) - 1$ und $g(x) = \sqrt{2} \cdot x + 2$

$f'(x) = 2\cos(x)$ $\qquad g'(x) = \sqrt{2}$

$2\cos(x) \overset{!}{=} \sqrt{2}$

$\cos(x) = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \longrightarrow x = \frac{\pi}{4}$



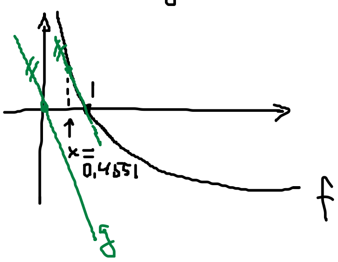
c) $f(x) = -\log_3(x)$ und $g(x) = -2x$

$f'(x) = -\frac{1}{\log(3) \cdot x}$ $\qquad g'(x) = -2$

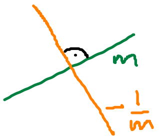
$\frac{d}{dx}\log_a(x) = \frac{1}{\log(a) \cdot x}$

$+\frac{1}{\log(3) \cdot x} \overset{!}{=} +2 \qquad | \cdot x$

$x = \underline{\frac{1}{2 \cdot \log(3)}} = 0.4551$



Aufgabe 4: Wie gross muss der Parameter a sein, damit sich die Funktionen f und g in einem rechten Winkel schneiden?



a) $f(x) = 2 + \frac{x}{5}$ und $g(x) = ax + 2$

$f'(x) = \frac{1}{5}$ $\qquad$ $g'(x) = a$

$m_f = \frac{1}{5} = -\frac{1}{m_g} = -\frac{1}{a}$ $\qquad$ $\frac{1}{5} = -\frac{1}{a}$

$\underline{a = -5}$

b) $f(x) = (x+3)(x+a)$ und $g(x) = -(x+3)(x-a)$ im fixen Schnittpunkt: Nullstelle $(-3, 0)$

$f(x) = x^2 + (3+a)x + 3a$ $\qquad$ $g(x) = -x^2 - (3-a)x + 3a$

$f'(x) = 2x + 3 + a$ $\qquad$ $g'(x) = -2x - (3-a) = -2x - 3 + a$

$m_f = 2\cdot(-3) + 3 + a = a - 3$ $\qquad$ $m_g = -2\cdot(-3) - 3 + a = 3 + a$

$m_f = -\frac{1}{m_g} \Rightarrow a - 3 = -\frac{1}{3+a} \rightarrow (a+3)(a-3) = -1$

$a^2 - 9 = -1$

$a^2 = 8 \longrightarrow \underline{a = \pm\sqrt{8}}$

c) $f(x) = (x+2)(x+a)$ und $g(x) = x + 2$ im fixen Schnittpunkt: Nullstelle $(-2, 0)$

$f(x) = x^2 + (2+a)x + 2a$ $\qquad$ $g'(x) = 1$

$f'(x) = 2x + 2 + a$

$\rightarrow m_f = 2\cdot(-2) + 2 + a = a - 2$ $\qquad$ $m_g = 1$

$m_f = a - 2 = -\frac{1}{m_g} = -\frac{1}{1} \rightarrow a - 2 = -1$

$\underline{a = 1}$

d) Optional: Bestimmen Sie a für den Fall c), jedoch für den variablen Schnittpunkt

$f(x) = x^2 + (2+a)x + 2a \overset{!}{=} x + 2 = g(x)$

$x^2 + (2+a-1)x + 2a - 2 = 0$

$x^2 + (1+a)x + 2(a-1) = 0$

$A = 1,\ B = 1+a,\ C = 2(a-1)$ $\qquad$ $D = B^2 - 4AC = (1+a)^2 - 4\cdot 1\cdot 2(a-1)$

$= 1 + 2a + a^2 - 8a + 8$

$= a^2 - 6a + 9$

$D = (a-3)^2$

$\sqrt{D} = a - 3$

$x_{1,2} = \frac{-1 - a \pm (a-3)}{2}$

$x_1 = \frac{-1 \cancel{-a} \cancel{+a} - 3}{2} = -2$

$x_2 = \frac{-1 - a - a + 3}{2} = \frac{2 - 2a}{2} = \underline{1 - a}$

↳ x-Koord. des var. Schnittpunkts

$f'(x) = 2x + 2 + a \rightarrow f'(1-a) = 2\cdot(1-a) + 2 + a = 2 - 2a + 2 + a = 4 - a = m_f$

$g'(x) = 1 = m_g$

$\hookrightarrow m_f = -\frac{1}{m_g} \Rightarrow 4 - a = -1 \rightarrow \underline{a = 5}$