# Rationale Funktionen

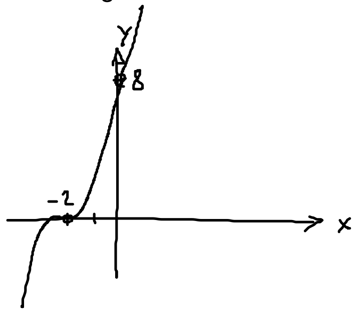
**Aufgabe 1:** Ermitteln Sie für die folgenden Polynomfunktionen die Nullstellen, Achsabschnitte und den Grad der Funktion. Skizzieren Sie dann den ungefähren Verlauf des Funktionsgraphen aufgrund dieser Informationen.

a) $f(x) = (x+2)^3$

Nullstelle : $x = -2$ (dreifach)

Grad: 3

$f(0) = 2^3 = 8$

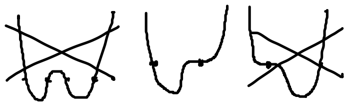


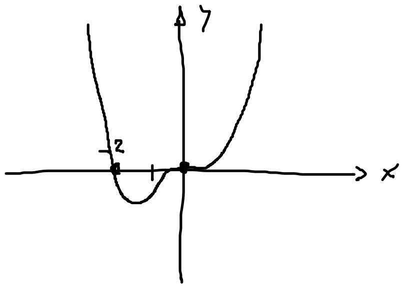

b) $f(x) = x^4 + 2x^3 = x^3(x+2)$

Grad: 4

Nullstellen : $x = 0$ (3-fach) , $x = -2$

$f(0) = 0$





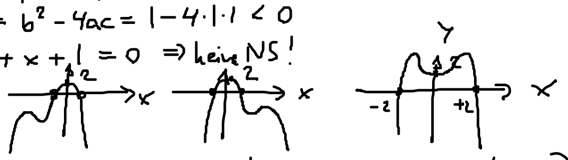

c) $f(x) = -\frac{1}{2}(x^2+x+1)\underbrace{(x^2-4)}_{(x-2)(x+2)}$ $f(0) = -\frac{1}{2}(0+0+1)(0-4) = 2$

$a = 1$, $b = 1$, $c = 1$

$D = b^2 - 4ac = 1 - 4 \cdot 1 \cdot 1 < 0$

$\rightarrow x^2 + x + 1 = 0 \Rightarrow$ keine NS!

Grad: 4




d) $f(x) = -\frac{1}{2}x^2 + 4x - 4$ $a = -\frac{1}{2}$, $b = 4$, $c = -4$ $D = b^2 - 4ac = 16 - 4 \cdot (-\frac{1}{2}) \cdot (-4)$

$= 16 - 8 = 8$

$\sqrt{D} = \sqrt{8}$

Grad: 2

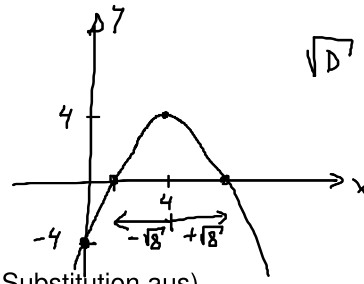
$x_{1,2} = \frac{-4 \pm \sqrt{8}}{-1} = 4 \pm \sqrt{8}$ (Nullstellen)

$f(x) = (x - (4+\sqrt{8}))(x - (4-\sqrt{8}))$

$\sqrt{8} = 2\sqrt{2} \approx 2.8$

$f(0) = -4$

$f(4) = -\frac{1}{2}4^2 + 4 \cdot 4 - 4 = -8 + 16 - 4 = 4$




e) $f(x) = 2x^4 - 5x^2 + 2$ (Tipp: Führen Sie eine Substitution aus)

Grad: 4

$u = x^2 \rightarrow f(u) = 2u^2 - 5u + 2$ $a = 2$, $b = -5$, $c = 2$ $\rightarrow D = 25 - 4 \cdot 2 \cdot 2$

$= 25 - 16 = 9$

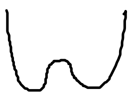
$\sqrt{D} = 3$

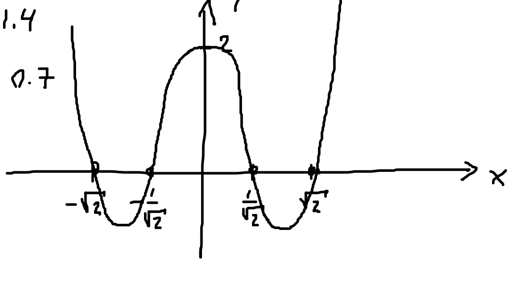
$u_{1,2} = \frac{5 \pm 3}{4} = 1.25 \pm 0.75$ ↗ 2 ↘ 0.5

$u_1 = 2 = x^2 \rightarrow x_{1,2} = \pm\sqrt{2} \approx \pm 1.4$

$u_2 = \frac{1}{2} = x^2 \rightarrow x_{3,4} = \pm\frac{1}{\sqrt{2}} \approx \pm 0.7$

$f(0) = 2$





**Aufgabe 2:** Berechnen Sie die folgenden Summen und Differenzen. Bringen Sie den Zähler in Normalform. Der Nenner darf in Produktform bleiben.

a) $\frac{x}{x^2+3x+2} + \frac{x-3}{x^2-x-2} = \frac{x}{(x+1)(x+2)} + \frac{x-3}{(x+1)(x-2)} = \frac{x(x-2)+(x-3)(x+2)}{(x+1)(x+2)(x-2)}$

$x^2+3x+2 = (x+1)(x+2)$

$+1+2=3 \quad (+1)\cdot(+2)=2$

$x^2-x-2 = (x+1)(x-2)$

$1-2=-1 \quad 1\cdot(-2)=-2$

kgV $= (x+1)(x+2)(x-2)$

$= \frac{x^2-2x+x^2-x-6}{(x+1)(x+2)(x-2)}$

$= \frac{2x^2-3x-6}{(x+1)(x+2)(x-2)}$

b) $\frac{x}{x^2-6x+9} - \frac{x-3}{-3+x(x-2)} = \frac{x(x+1)}{(x-3)^2(x+1)} - \frac{(x-3)(x-3)}{(x-3)^2(x+1)}$

$x^2-6x+9 = (x-3)^2$

$-3+x(x-2) = x^2-2x-3 = (x+1)(x-3)$

$+1-3=-2 \quad (+1)\cdot(-3)=-3$

kgV $= (x-3)^2(x+1)$

$= \frac{x^2+x-(x^2-6x+9)}{(x-3)^2(x+1)}$

$= \frac{7x-9}{(x-3)^2(x+1)}$

c) $\frac{1}{x-1} - \frac{1}{x+1} + 1$

$= \frac{(x+1)}{(x-1)(x+1)} - \frac{(x-1)}{(x-1)(x+1)} + \frac{(x-1)(x+1)}{(x-1)(x+1)}$

$= \frac{\cancel{x}+1-\cancel{x}+1+(x-1)(x+1)}{(x-1)(x+1)} = \frac{2+x^2-1}{x^2-1} = \frac{x^2+1}{x^2-1}$

d) $\frac{x^2+x+1}{x^2+\frac{3}{4}x-\frac{5}{8}} - \frac{x+\frac{1}{2}}{x^2-\frac{50}{32}} = \frac{(x^2+x+1)(x-\frac{5}{4}) - \overbrace{(x+\frac{1}{2})(x-\frac{1}{2})}^{x^2-\frac{1}{4}}}{(x-\frac{1}{2})(x+\frac{5}{4})(x-\frac{5}{4})}$

$x^2+\frac{3}{4}x-\frac{5}{8} \quad a=1,\ b=\frac{3}{4},\ c=-\frac{5}{8} \rightarrow D=b^2-4ac = \frac{9}{16}+4\cdot1\cdot\frac{5}{8} = \frac{9}{16}+\frac{20}{8}$

$D = \frac{49}{16}$

$\sqrt{D} = \frac{7}{4}$

$x_{1,2} = \frac{-\frac{3}{4} \pm \frac{7}{4}}{2} = -\frac{3}{8} \pm \frac{7}{8}$ $\quad x_1 = +\frac{1}{2}, \quad x_2 = -\frac{5}{4}$

$\rightarrow (x-\frac{1}{2})(x+\frac{5}{4}) = x^2+\frac{5}{4}x-\frac{2}{4}x-\frac{5}{8} = x^2+\frac{3}{4}x-\frac{5}{8}$ ✓

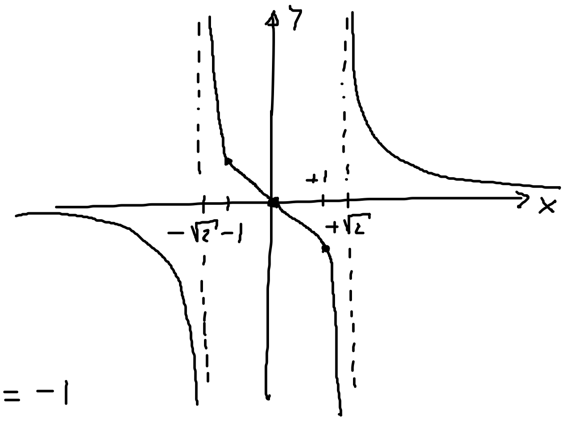
$x^2-\frac{50}{32} = x^2-\frac{25}{16} = x^2-(\frac{5}{4})^2 = (x-\frac{5}{4})(x+\frac{5}{4})$

$\Rightarrow$ kgV: $(x-\frac{1}{2})(x+\frac{5}{4})(x-\frac{5}{4})$

$= \frac{x^3+\cancel{x^2}+x-\frac{5}{4}x^2-\frac{5}{4}x-\frac{5}{4}-\cancel{x^2}+\frac{1}{4}}{(\cdots)(\cdots)(\cdots)} = \frac{x^3-\frac{5}{4}x^2-\frac{1}{4}x-1}{(x-\frac{1}{2})(x+\frac{5}{4})(x-\frac{5}{4})}$

**Aufgabe 3:** Zeichnen Sie die Graphen der folgenden Funktionen und bestimmen Sie die Asymptoten mit Hilfe der Polynomdivision. Ermitteln Sie jeweils auch die Nullstellen und Unstetigkeiten.

a) $f(x) = \dfrac{x}{x^2 - 2}$




Nullstelle für $x = 0$

Unstetigkeit für $x^2 - 2 = 0 \rightarrow x = \pm\sqrt{2}$

Grad: $1 - 2 = -1$ (Hyperbel)

$f(0) = 0$

Polynomdivision: $x : (x^2 - 2) = \frac{1}{x} + \text{Rest } \frac{2}{x}$

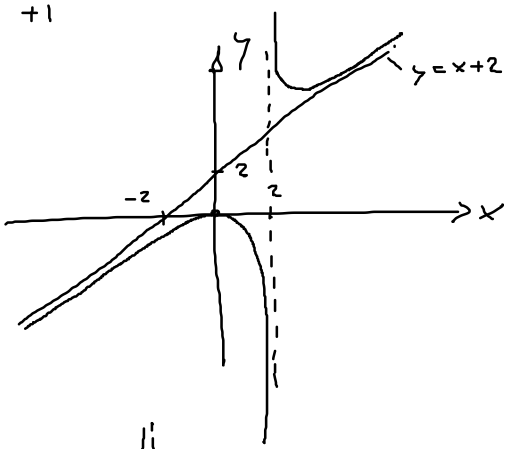
$-x + \frac{2}{x}$

$\frac{2}{x}$

$f(+1) = \frac{1}{1-2} = -1$

$f(-1) = \frac{-1}{1-2} = +1$

b) $f(x) = \dfrac{x^2}{x - 2}$




Grad: $2 - 1 = +1$

Nullstellen: $x = 0$ (zweifach)

Unstetigkeit: $x = 2$

Polynomdivision:

$x^2 : (x - 2) = x + 2$ Rest 4

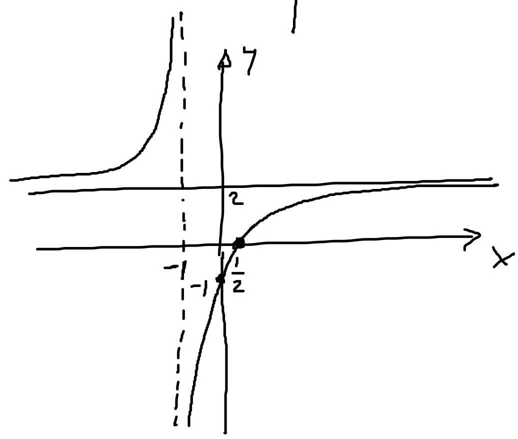
$-x^2 + 2x$

$2x$

$-2x + 4$

$4$

c) $f(x) = \dfrac{4x - 2}{2x + 2}$




Grad: $1 - 1 = 0$

Nullstelle: $4x - 2 = 0 \quad x = \frac{1}{2}$

Unstetigkeit: $2x + 2 = 0 \quad x = -1$

Polynomdivision:

$4x - 2 : (2x + 2) = 2$ Rest $-6$

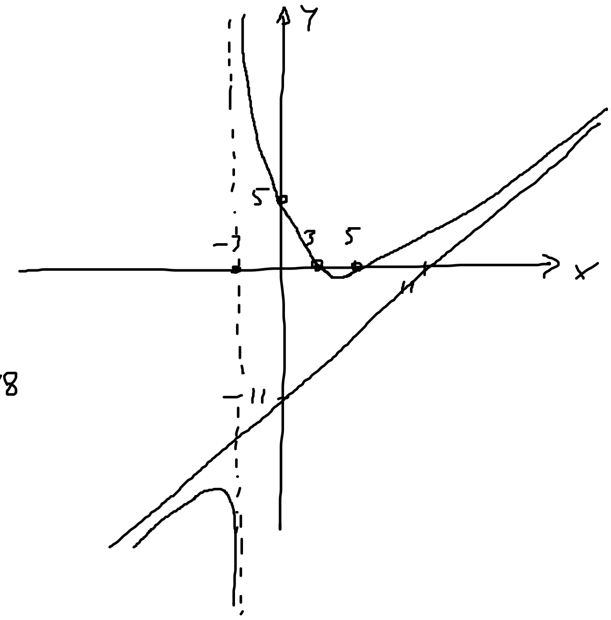
$-4x \mp 4$

$-6$

$f(0) = \frac{-2}{+2} = -1$

$-3 - 5 = -8$

d) $f(x) = \dfrac{x^2 - 8x + 15}{x + 3}$ $\leftarrow (-3)\cdot(-5) = +15$




Grad: $2 - 1 = 1$

Nullstellen: $(x - 3)(x - 5) \rightarrow x = 3, 5$

Unstetigkeit: $x = -3$

$f(0) = \frac{15}{3} = 5$

Polynomdivision:

$x^2 - 8x + 15 : (x + 3) = x - 11$ Rest 48

$-x^2 \mp 3x$

$-11x$

$+11x + 33$

$48$

e) $f(x) = \dfrac{x^3 - 1}{x^2 - x}$ $\quad f(2) = \dfrac{8-1}{4-2} = \dfrac{7}{2} = 3.5 \quad f(-1) = \dfrac{-1-1}{1-(-1)} = \dfrac{-2}{2} = -1$

Grad: $3-2 = 1$

Nullstellen: $x^3 - 1 = 0 \rightarrow x^3 = +1 \rightarrow x = 1$ (3-fach)

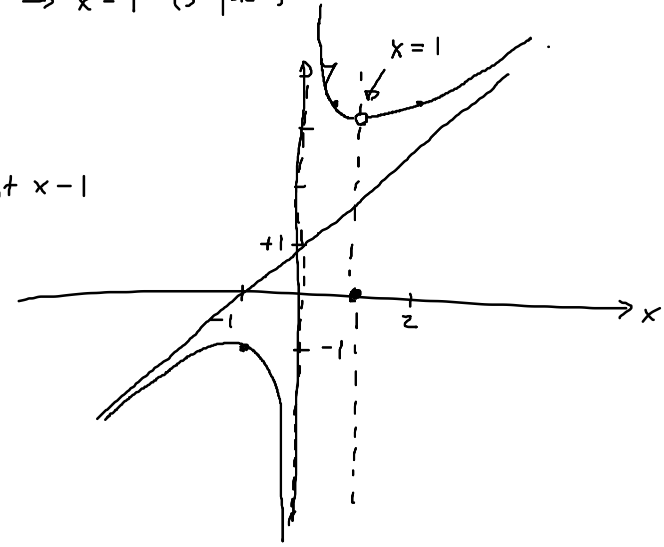
Unstetigkeit: $x(x-1) = 0 \rightarrow x = 0, 1$

Polynomdivision:

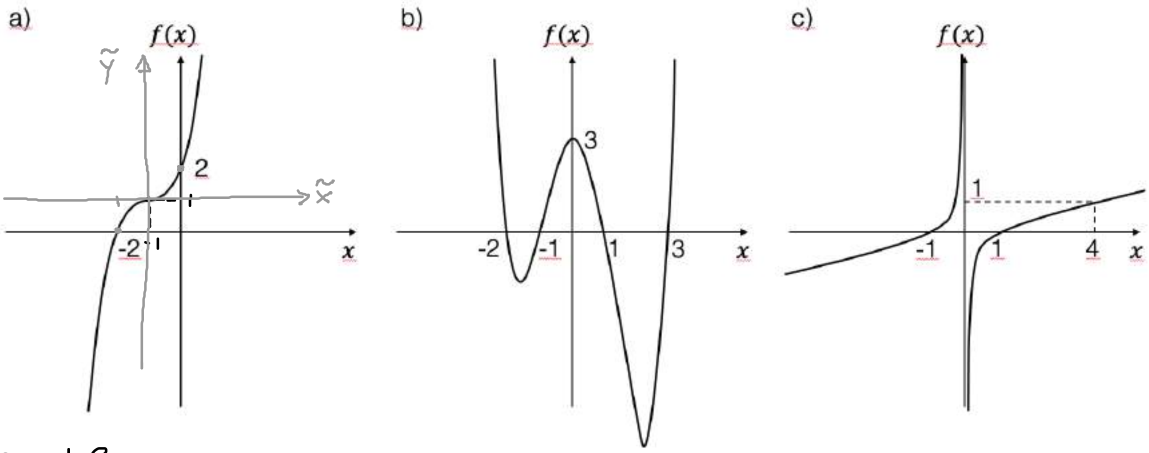
$$x^3 - 1 : (x^2 - x) = \underline{x + 1} \quad \text{Rest } x - 1$$

$$\begin{array}{l} \underline{-x^3 + x^2} \\ \quad x^2 \\ \quad \underline{-x^2 + x} \\ \qquad x - 1 \end{array}$$

$$f\left(\tfrac{1}{2}\right) = \frac{\frac{1}{8} - 1}{\frac{1}{4} - \frac{1}{2}} = \frac{-\frac{7}{8}}{-\frac{2}{8}} = \frac{7}{2} = 3.5$$



**Aufgabe 4:** Finden Sie die Normalform der folgenden Funktionen.

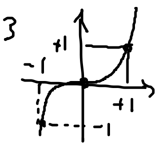



a) Grad 3

Nullstelle: $x = -2$

Achsabschnitt: $f(0) = 2$

$y = x^3$




$\tilde{y}(\tilde{x}) = \tilde{x}^3 \rightsquigarrow y - 1 = (x+1)^3 \to \underline{y = (x+1)^3 + 1}$

$x = \tilde{x} - 1 \to \tilde{x} = x + 1$

$y = \tilde{y} + 1 \to \tilde{y} = y - 1$

$y(-2) = (-2+1)^3 + 1 = 0$ ✓

$y(0) = (0+1)^3 + 1 = 2$ ✓

$y = x^3 + 3x^2 + 3x + 1^3 + 1 = \underline{x^3 + 3x^2 + 3x + 2}$

b) Grad 4, 4 Nullstellen: $x = -2, -1, 1, 3$ Achsabschnitt: $y(0) = 3$

$y = a(x+2)(x+1)(x-1)(x-3)$

$y(0) = a \cdot 2 \cdot 1 \cdot (-1) \cdot (-3) = 3$

$6a = 3$

$a = \frac{1}{2}$

$$
\begin{aligned}
y(x) &= \tfrac{1}{2}(x+2)(x+1)(x-1)(x-3) \\
&= \tfrac{1}{2}(x+2)(x^2-1)(x-3) \\
&= \tfrac{1}{2}(x^3 - x + 2x^2 - 2)(x-3) \\
&= \tfrac{1}{2}(x^4 - x^2 + 2x^3 - 2x - 3x^3 + 3x - 6x^2 + 6) \\
&= \tfrac{1}{2}(x^4 - x^3 - 7x^2 + x + 6)
\end{aligned}
$$

$$\underline{y(x) = \tfrac{1}{2}x^4 - \tfrac{1}{2}x^3 - \tfrac{7}{2}x^2 + \tfrac{1}{2}x + 3}$$

c) Asymptote: $y = \frac{1}{4}x$

Unstetigkeit für $x = 0$

Grad: $2 - 1 = 1$

Nullstellen: $x = -1, +1$

$y(x) = \dfrac{(x+1)(x-1)}{4x} = \underline{\dfrac{x^2-1}{4x}}$

Polynomdivision: $x^2 - 1 : 4x = \underline{\frac{1}{4}x}$ Rest $-1$

$\underline{-x^2}$

$-1$